

Effect of q -deformation in the NJL gap equation

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We obtain a q -deformed algebra version of the Nambu–Jona-Lasinio model gap equation. In this framework we discuss some hadronic properties such as the dynamical mass generated for the quarks, the pion decay constant and the phase transition present in this model.

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The concept of symmetry is of fundamental importance in physics; the breaking of a symmetry and its associated phase transition are universal phenomena appearing in many branches in physics, such as nuclear and solid state physics, although the broken symmetries and the physical systems involved are quite different. Y. Nambu was the first to realize this universal aspect of dynamical symmetry breaking [1]. The Nambu-Jona-Lasinio (NJL) model is very adequate to study the breaking of chiral symmetry and the generation of a dynamical mass for the quarks due to the appearance of condensates.

On another side, in the last few years the study of q -deformed algebras turned out to be a fertile area of research. The use of q -deformed algebra in the description of some many-body systems has lead to the appearance of new features when compared to the non-deformed case [2]. In particular, it seems to be a very elegant framework to describe perturbations from some underlying symmetry structure. From the many applications of q -deformation ideas existing in the literature, ranging from optics to particle physics, we would like to pinpoint three of them: the investigation of the behavior of the second order phase transition in a q -deformed Lipkin model [3], the good agreement with the experimental data obtained through a κ -deformed Poincaré phenomenological fit to the dynamical mass and rotational and radial excitations of mesons [4], and the purely $su_q(2)$ -based mass formula for quarks and leptons developed by using an inequivalent representation [5].

It sounds therefore reasonable to study the influence of the q -deformation on the mass generation mechanism due to the breaking of chiral symmetry. To be definite, in this work we intend to investigate the effects of the q -deformation on the phase transition of the NJL model, stimulated by an analogy with q -deformed Lipkin model, where the phase transition is smoothed down when the Lipkin Hamiltonian is deformed [3].

Recently, the thermodynamical properties of a free quantum group fermionic system with two “flavors” were studied [6]. In particular, it was given there a $su_q(N)$ -covariant representation of the fermionic algebra for arbitrary N in terms of ordinary creation and annihilation operators. The $su_q(2)$ -covariant algebra is given by the following relations

$$\{\psi_1, \bar{\psi}_1\} = 1 - (1 - q^{-2}) \bar{\psi}_2 \psi_2 \quad \{\psi_2, \bar{\psi}_2\} = 1, \quad (1)$$

$$\psi_1 \psi_2 = -q \psi_2 \psi_1 \quad \bar{\psi}_1 \psi_2 = -q \psi_2 \bar{\psi}_1, \quad (2)$$

$$\{\psi_1, \psi_1\} = 0 \quad \{\psi_2, \psi_2\} = 0. \quad (3)$$

The usual $su(2)$ covariant fermionic algebra is recovered when $q = 1$. Later, the pure nuclear pairing force version of the Bardeen-Cooper-Schrieffer (BCS) many-body formalism [7] was extended in such a way to replace the usual fermions by quantum group covariant ones satisfying appropriate anticommutation relations for a $su_q(N)$ -fermionic algebra [8]. Using the $su_q(2j+1)$ -covariant representation of the fermionic algebra, a q -covariant form of the BCS approximation was constructed and the q -analog to the BCS equations along with the quantum gap equation was derived. The quantum gap was shown to depend explicitly on the deformation parameter and it is reduced as the deformation increases.

The Nambu–Jona-Lasinio model was first introduced to describe the nucleon-nucleon interaction via a four-fermion contact interaction. Later, the model was extended to quark degrees of freedom becoming an effective model for quantum chromodynamics (QCD).

The Lagrangian of the NJL model is given by

$$\mathcal{L}_{NJL} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \mathcal{L}_{int}, \quad (4)$$

$$\mathcal{L}_{int} = G \left[(\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \tau \psi)^2 \right]. \quad (5)$$

Linearizing the above interaction in a mean field approach, the last term does not contribute if the vacuum is parity and Lorentz invariant. The Lagrangian with the linearized interaction is then

$$\mathcal{L}_{NJL} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + 2G \langle \bar{\psi} \psi \rangle \bar{\psi} \psi. \quad (6)$$

Regarding this Lagrangian as a Dirac Lagrangian for massive quarks we obtain a dynamical mass for the quarks

$$m = -2G \langle \bar{\psi} \psi \rangle, \quad (7)$$

where $\langle \bar{\psi} \psi \rangle$ is the vacuum expectation value of the scalar density $\bar{\psi} \psi$, representing the quark condensates. Eq. (7) describes how the dynamical mass is generated with the appearance of the quark condensates. The quarks are massless if the condensate vanishes.

We now turn to the q -deformation of the NJL gap equation. Following [6,8] we write the creation and annihilation operators of the $su_q(2j+1)$ fermionic algebra as,

$$A_{jm} = a_{jm} \prod_{i=m+1} \left(1 + Q a_{ji}^\dagger a_{ji} \right), \quad (8)$$

$$A_{jm}^\dagger = a_{jm}^\dagger \prod_{i=m+1} \left(1 + Q a_{ji}^\dagger a_{ji} \right), \quad (9)$$

where $Q = q^{-1} - 1$, $j = 1/2$ and $m = \pm 1/2$. The first consequence of the above deformation is that only the operators corresponding to $m = -\frac{1}{2}$ are modified. Since in the NJL model we deal with quarks (anti-quarks) creation and annihilation operators, this feature is important because only negative helicity quarks (anti-quarks) operators will be deformed. Explicitly, we have

$$A_- = a_- \left(1 + Q a_+^\dagger a_+ \right), \quad A_-^\dagger = a_-^\dagger \left(1 + Q a_+^\dagger a_+ \right), \quad (10)$$

$$A_+ = a_+ \quad , \quad A_+^\dagger = a_+^\dagger, \quad (11)$$

where $+$ ($-$) stands for the positive (negative) helicity. In a sense, we are embedding the chiral symmetry breaking effects in the operators' definition.

We are now in position to obtain the deformed gap equation by introducing a BCS-like vacuum and proceeding similarly to the standard Bogoliubov-Valatin approach [9]. The q -deformed BCS vacuum reads

$$|NJL\rangle = \prod_{\mathbf{p}, s=\pm 1} [\cos \theta(p) + s \sin \theta(p) B^\dagger(\mathbf{p}, s) D^\dagger(-\mathbf{p}, s)] |0\rangle \quad (12)$$

and the quark fields are expressed in terms of q -deformed creation and annihilation operators as

$$\psi_q(x, 0) = \sum_s \int \frac{d^3 p}{(2\pi)^3} [B(\mathbf{p}, s) u(\mathbf{p}, s) e^{i\mathbf{p} \cdot \mathbf{x}} + D^\dagger(\mathbf{p}, s) v(\mathbf{p}, s) e^{-i\mathbf{p} \cdot \mathbf{x}}]. \quad (13)$$

The q -deformed quark and anti-quark creation and annihilation operators B , B^\dagger , D , and D^\dagger , are expressed in terms of the non-deformed ones according to Eqs. (10) and (11)

$$B_- = b_- \left(1 + Q b_+^\dagger b_+ \right), \quad B_-^\dagger = b_-^\dagger \left(1 + Q b_+^\dagger b_+ \right), \quad (14)$$

$$D_- = d_- \left(1 + Q d_+^\dagger d_+ \right), \quad D_-^\dagger = d_-^\dagger \left(1 + Q d_+^\dagger d_+ \right), \quad (15)$$

$$B_+ = b_+ \quad , \quad B_+^\dagger = b_+^\dagger, \quad (16)$$

$$D_+ = d_+ \quad , \quad D_+^\dagger = d_+^\dagger, \quad (17)$$

(in the above equations we simplified the notation: $B(\mathbf{p}, s) \rightarrow B_s$, $b(\mathbf{p}, s) \rightarrow b_s$, etc.). We would like to stress that, as discussed in Ref. [8], the deformed vacuum differs from the non-deformed one only by a phase and, therefore, the effects of the deformation comes solely from the modified field operators. Additionally, the q -deformed NJL

Lagrangian, constructed using ψ_q instead of ψ , is invariant under the quantum group $SU_q(2)$ transformations. This can be seen by using the two-dimensional representation of the $SU_q(2)$ unitary transformation given in Ref. [6].

The deformed gap equation is

$$m = -2G \langle \bar{\psi}\psi \rangle_q, \quad (18)$$

where $\langle \bar{\psi}\psi \rangle_q$ is the q -deformed condensate calculated using the BCS-like vacuum, Eq. (12), and Eq. (13),

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_q &= \langle NJL | \bar{\psi}_q \psi_q | NJL \rangle \\ &= \langle \bar{\psi}\psi \rangle + \langle NJL | \mathcal{Q} | NJL \rangle, \end{aligned} \quad (19)$$

where $\langle \bar{\psi}\psi \rangle$ is the non-deformed condensate and $\langle NJL | \mathcal{Q} | NJL \rangle$ represents all non-vanishing matrix elements arising from the q -deformation of the quark fields. The contribution of these q -deformed matrix elements is

$$\langle NJL | \mathcal{Q} | NJL \rangle = Q \int \frac{d^3p}{(2\pi)^3} [\sin 2\theta(p) - \sin 2\theta(p) \cos 2\theta(p)]. \quad (20)$$

The calculation of the deformed condensate will be performed in a similar way as in the usual case [10]. It requires also a regularization procedure since the NJL interaction is not perturbatively renormalizable. For reasons of simplicity a non-covariant trimomentum cutoff is applied arising

$$\langle \bar{\psi}\psi \rangle_q = -\frac{3m}{\pi^2} \left[\left(1 - \frac{Q}{2}\right) \int_0^\Lambda dp \frac{p^2}{\sqrt{\mathbf{p}^2 + m^2}} + \frac{Q}{2} \int_0^\Lambda dp \frac{p^3}{\mathbf{p}^2 + m^2} \right] \quad (21)$$

for each quark flavor. At this point we see that the dynamical mass is again given by a self-consistent equation since the condensate depends also on the mass. Inserting Eq. (21) into Eq. (18) we obtain the deformed NJL gap equation

$$m = \frac{2Gm}{\pi^2} \left[\left(1 - \frac{Q}{2}\right) \int_0^\Lambda dp \frac{p^2}{\sqrt{\mathbf{p}^2 + m^2}} + \frac{Q}{2} \int_0^\Lambda dp \frac{p^3}{\mathbf{p}^2 + m^2} \right]. \quad (22)$$

It is easy to see that for $Q = 0$ ($q = 1$), we recover the NJL gap equation in its more familiar form

$$m = \frac{2Gm}{\pi^2} \int_0^\Lambda dp \frac{p^2}{\sqrt{\mathbf{p}^2 + m^2}} + m_0, \quad (23)$$

where m_0 appears only if we consider the current quark mass term $\mathcal{L}_{mass} = -m_0 \bar{\psi}\psi$ in the NJL Lagrangian Eq. (4).

The pion decay constant is calculated from the vacuum to one pion axial vector current matrix element, which, in the simple 3D non-covariant cutoff we are using [9], is given by

$$f_\pi^2 = N_c m^2 \int_0^\Lambda \frac{d^3p}{(2\pi)^3} \frac{1}{(\mathbf{p}^2 + m^2)^{3/2}}, \quad (24)$$

for each quark color. The deformed calculation of f_π is performed directly by substituting the dynamical mass in Eq. (24) from the one obtained in Eq. (22), instead of deforming the axial current in the calculation of its matrix element of between the vacuum and the one pion state.

As in the non-deformed case, the q -gap equation has non-trivial solutions when the coupling G exceeds a critical value G_{crit} related to the cutoff. Figure 1 depicts the sharp phase transition at $G = G_{crit}$ separating the Wigner-Weyl and Nambu-Goldstone phases, corresponding to different realizations of chiral symmetry.

Figure 1 also shows the deformed condensate values as a function of q . We can see the enhancement of the condensate's value, due to presence of the q -deformation. The dynamical mass is accordingly modified through the deformed gap equation (22), as can be seen in Table I, along with the corresponding values of the pion decay constant, f_π . The behavior of the condensate around the critical coupling, G_c , is similar for both deformed and non-deformed cases, meaning that the adopted procedure to q -deform the underlying $su(2)$ algebra in a two flavor NJL model does not change the behavior of the phase transition around G_c . Table II presents the behavior of the coupling constant for two typical dynamical mass values for different q 's. The analysis of this table shows that the coupling constant G decreases with q , for a given value of the dynamical mass, meaning that to acquire a given mass we need a weaker coupling when the algebra is deformed. This indicates that the deformation of the $su(2)$ algebra incorporates effects

the NJL interaction which are propagated to the physical quantities such as the condensate, the dynamical mass and the pion decay constant. The formalism developed in [6,8] allow q -values smaller than one (which corresponds to $Q > 0$). It is worth to mention that in this case the q -deformation effect goes in the opposite direction, namely, the condensate value and the dynamical mass decrease for $q < 1$.

To summarize, the main objective of this work was to analyze the influence of the q -deformation in the NJL model. We studied the deformation of the underlying $su(2)$ algebra in a two flavor version of the model and investigated an important feature of the $su(2)$ chiral symmetry breaking, namely the dynamical mass generation, through the incorporation of helicity non-conserving terms directly in the fermionic operators. The main effect of the q -deformation is to effectively enhance the coupling strength of the NJL four fermion interaction, leading to an increasing in the value of the quark condensate. The dynamical mass, which is related to the presence of the condensate, is correspondingly increased. We also looked closely at the behavior of the phase transition around the critical point, which is still sharp, meaning that the new contributions arising from the deformation of the condensate do not play the role of explicit chiral symmetry breaking terms [10].

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	q = 1.0	q = 1.1	q = 1.2	q = 1.3
Mass [MeV]	365	375	384	392
f_π [MeV]	92.0	92.4	92.9	93.3

TABLE I. Mass and f_π for different values of q , for $\Lambda=600$ MeV, $G_c=4.57$ GeV $^{-2}$, and $G=6.53$ GeV $^{-2}$. The condensate was calculated for two flavors and three colors.

Mass [MeV]	$q = 1.0$	$q = 1.1$	$q = 1.2$	$q = 1.3$
300	6.04	5.98	5.94	5.90
350	6.42	6.35	6.28	6.23

TABLE II. Behavior of the coupling constant G at given values of the dynamical mass for different values of q . Cutoff and critical coupling are the same as in Table I.

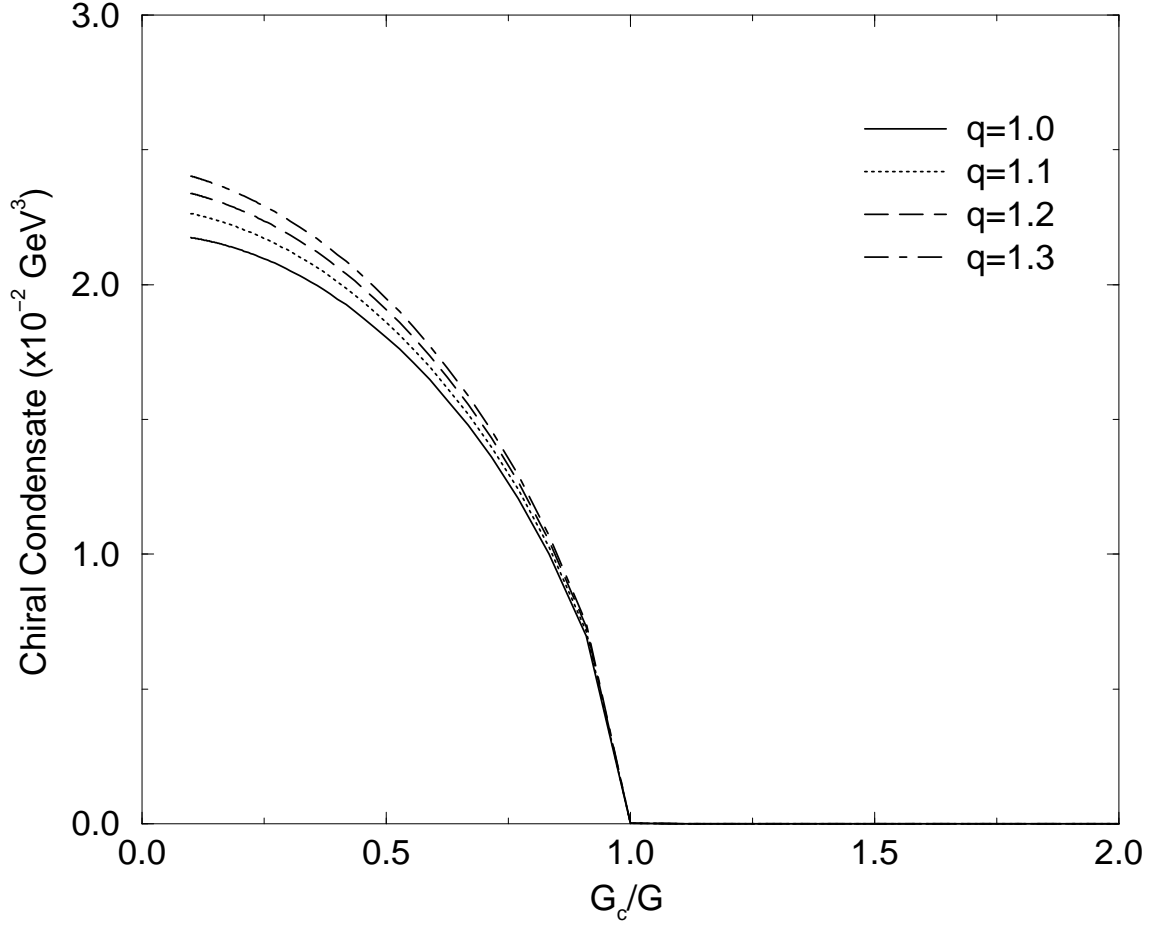


FIG. 1. Behavior of the phase transition for different values of q . The non-deformed situation corresponds to $q = 1$. In all curves $\Lambda = 600MeV$ and $m_0 = 0$.